# An update on the Eulerian formulation for the simulation of soft solids in fluids

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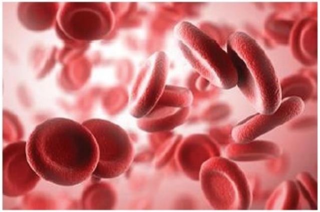
# Introduction and Motivation

- System of soft solids in a fluid are ubiquitous
  - Animal tissues, cell membrane.
- Studied for decades - arbitrary Lagrangian-Eulerian (ALE) method - Stiff solids.
- Need for a fully Eulerian approach with "*true solid constitutive laws*".
  - Eulerian Godunov method (Miller & Colella 2001) - unbounded domains
  - Reference Map Technique (RMT) (Kamrin et al. 2012)



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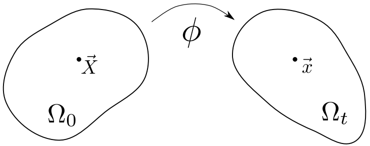
# Reference Map Technique




## Reference map

$$\vec{\xi}(\vec{x},t) = \vec{X}$$

$$\frac{D\vec{\xi}(\vec{x},t)}{Dt} = 0 \quad \Rightarrow \quad \frac{\partial\vec{\xi}(\vec{x},t)}{\partial t} + \vec{u}.\vec{\nabla}\vec{\xi}(\vec{x},t) = 0$$

## Deformation gradient

$$\mathbb{F}(\vec{X},t) = \partial\vec{x}/\partial\vec{X} = (\vec{\nabla}\vec{\xi}(\vec{x},t))^{-1}$$

[Valkov et al., *J. Appl. Mech.*, 82, 2015].

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# Governing equations for solids and fluids

## Momentum balance

$$\frac{\partial(\rho\vec{u})}{\partial t} + \nabla.(\rho\vec{u}\otimes\vec{u}) = \nabla.\underline{\underline{\sigma}}$$

## Mass balance

*Fluid:*

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}.(\vec{u}\rho) = 0 \qquad \Rightarrow \qquad \vec{\nabla}.\vec{u} = 0$$

*Solid:*

$$\rho = \rho_o(det(\mathbb{F}))^{-1} \qquad \Rightarrow \qquad det(\mathbb{F}) = 1$$

## Cauchy stress

*Newtonian fluid:*

$$\underline{\underline{\sigma}}^f = \mu\left[\left(\vec{\nabla}\vec{u}\right) + \left(\vec{\nabla}\vec{u}\right)^T\right] - \mathbb{1}\left(\vec{\nabla}P\right)$$

*Neo-Hookean solid:*

$$\underline{\underline{\sigma}}^s = 2(det\mathbb{F})^{-1}\mathbb{F}\frac{\partial\hat{\psi}(\mathbb{C})}{\partial\mathbb{C}}\mathbb{F}^T$$

$$\hat{\psi}(\mathbb{C}) = \mu[tr\mathbb{C} - 3] \qquad \Rightarrow \qquad \underline{\underline{\sigma}}^s = 2\mu^s[(\vec{\nabla}\vec{\xi})^T(\vec{\nabla}\vec{\xi})]^{-1}$$

[Valkov et al., *J. Appl. Mech.*, 82, 2015].

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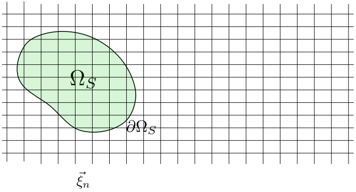
# Basic methodology

$\vec{\xi}$ defined only within the solid.




[Valkov et al., *J. Appl. Mech.*, 82, 2015].

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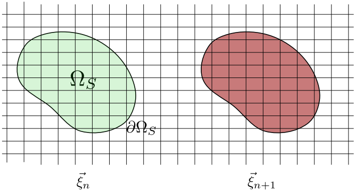
# Basic methodology

Procedure:

- Solve for $\vec{\xi}_{n+1}$ in $\Omega_S$.



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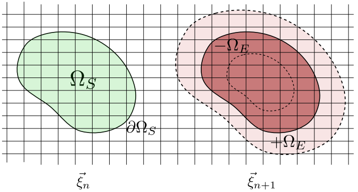
# Basic methodology

Procedure:

- Solve for $\vec{\xi}_{n+1}$ in $\Omega_S$.
- $\vec{\xi}_{n+1} \rightarrow$ Extrapolate outside $\Omega_S$.



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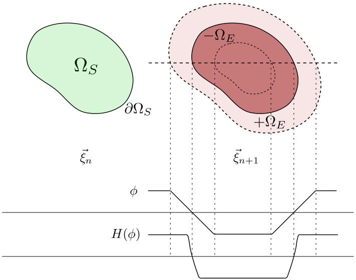
# Basic methodology

Procedure:

- Solve for $\vec{\xi}_{n+1}$ in $\Omega_S$.
- $\vec{\xi}_{n+1} \rightarrow$ Extrapolate outside $\Omega_S$.
- $\vec{\xi}_{n+1} \rightarrow$ Construct $\phi \rightarrow$ Construct $H(\phi)$.



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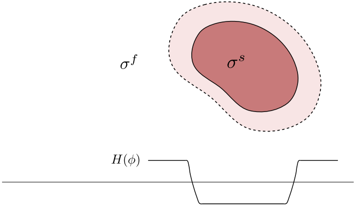
- Compute $\underline{\underline{\sigma}}^s, \underline{\underline{\sigma}}^f$.



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Procedure:

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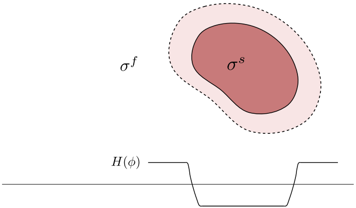
- $\vec{\xi}_{n+1} \rightarrow$ Extrapolate outside $\Omega_S$.
- $\vec{\xi}_{n+1} \rightarrow$ Construct $\phi \rightarrow$ Construct $H(\phi)$.
- Compute $\underline{\underline{\sigma}}^s, \underline{\underline{\sigma}}^f$.
- $\underline{\underline{\sigma}} \leftarrow$ Blend $f(H(\phi), \underline{\underline{\sigma}}^s, \underline{\underline{\sigma}}^f)$
- $\underline{\underline{\sigma}} \rightarrow$ Compute $\vec{u} \rightarrow$ Project $\vec{\nabla}.\vec{u} = 0$.



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# Closure model

Fluid-Solid coupling is based on the "*one-fluid formulation*".

## Smoothed heaviside function

$$H(x) = \begin{cases} 0 & x \leq -w_T \\ \frac{1}{2}(1 + \frac{x}{w_T} + \frac{1}{\pi} sin(\frac{\pi x}{w_T})) & |x| < w_T \\ 1 & x \geq w_T \end{cases}$$

constructed based on the reinitialized level-set field.

## Mixture model

$$\underline{\underline{\sigma}} = H(\phi(\vec{x}, t))\underline{\underline{\sigma}}^f + (1 - H(\phi(\vec{x}, t)))\underline{\underline{\sigma}}^s$$

$$\rho = H(\phi(\vec{x}, t))\rho^f + (1 - H(\phi(\vec{x}, t)))\rho^s$$

Capability to handle Solid-solid and solid-wall contact.

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# Updates and major issues fixed

- Momentum conservation - conservative form of equations.

$$\frac{\partial \rho \vec{u}}{\partial t} + \vec{\nabla}.(\rho \vec{u}\vec{u}) = \vec{\nabla}.(\sum_i \underline{\underline{\sigma}}_i)$$

- Non-dissipative schemes - stress evaluation, convective flux evaluation.
  - Central difference schemes
  - No viscous damping in the solid
- Modified reference map equation - momentum consistent solid advection.
  - Improved robustness

$$\frac{\partial \vec{\xi}(\vec{x},t)}{\partial t} + H(\psi)\vec{u}.\vec{\nabla}\vec{\xi}(\vec{x},t) = 0$$

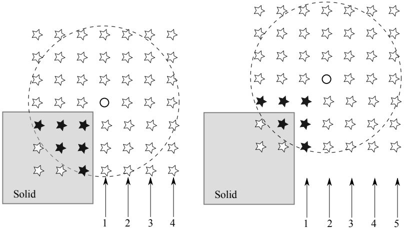
- Least-squares based extrapolation method.
- Collocated grid.

[Jain & Mani, *CTR Annual Research Briefs*, 2017]

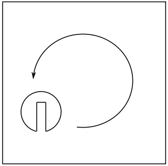
# Least-square method for extrapolation

**Locally linear approximation**

$$\xi = ax + by + c$$




| | **PDE** | **least-squares** |
|---|---|---|
| **Error** $(L_2)$ | $8.32 \times 10^{-4}$ | $6.72 \times 10^{-9}$ |
| **Cost** | $\approx$ 1550ms | $\approx$ 100ms |



Zalesak disk

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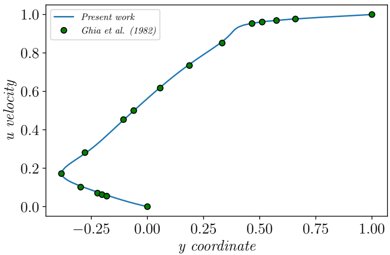
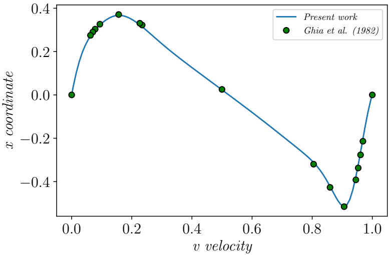
# Lid-driven cavity

- $Re = 1000$
- 100x100 grid.



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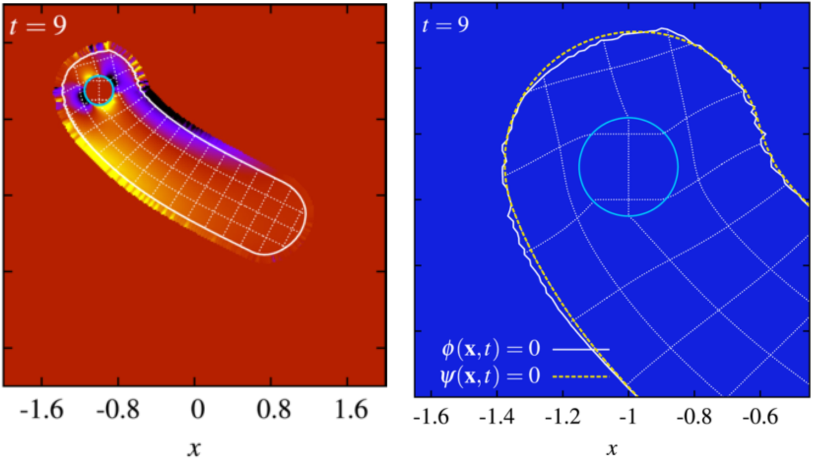
# Problems with advecting level-set field




[Valkov et al., 2015]

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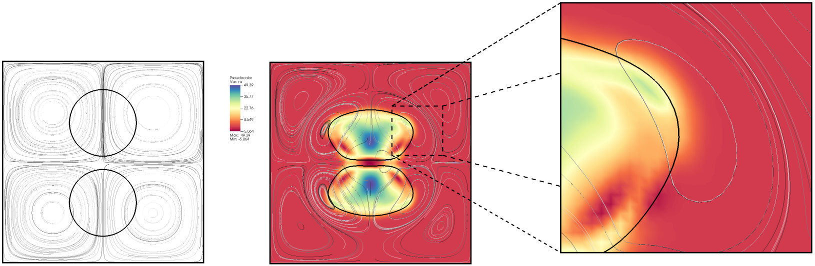
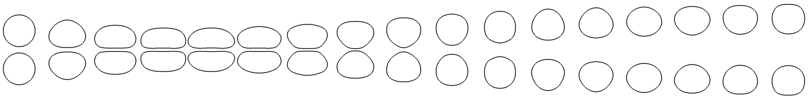
# Two solids in a Taylor-Green Vortex



*Initial field* *After collision* *Smooth interface*



**Modified reference map advection equation**

$$\frac{\partial \vec{\xi}(\vec{x},t)}{\partial t} + H(\psi)\vec{u}.\vec{\nabla}\vec{\xi}(\vec{x},t) = 0$$

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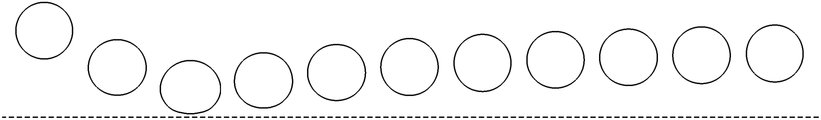
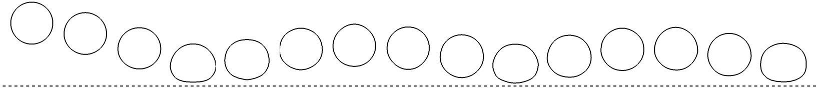
# Simulations of a solid in a fluid with wall contact

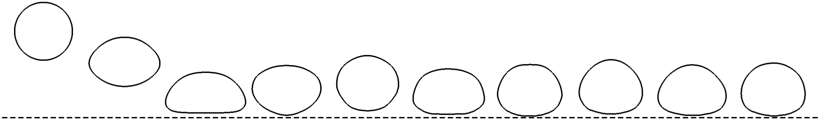
*Collision in microgravity:*



*Collision in gravity - Stiff solid:*



*Collision in gravity - Soft solid:*



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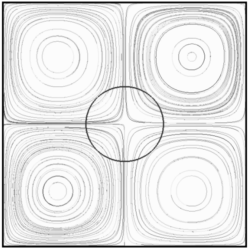
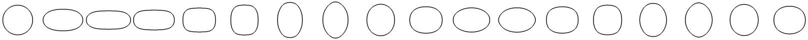
# Conservative vs Non-conservative formulation



*Initial state*

*Non-conservative formulation:*

*Conservative formulation:*



# Conclusion

*We here presented:*

- An incompressible fully Eulerian formulation for soft solids in fluids using an approximate Projection method.
- Improvements to the previously proposed Reference Map Technique:
  - Momentum conserving formulation.
  - Non-dissipative schemes for flux computation - better KE conservation.
  - Momentum consistent solid advection.
  - An accurate and cost effective extrapolation procedure.
  - Use of collocated grids.

*Future work:*

- Analytical results for a quantitative comparison of fully coupled systems.
- An implicit formulation.
- Multigrid solver to speed up the poisson solution.

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# THANK YOU

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*Ken Kamrin, MIT*

***Reference:***
Jain, S, S, & Mani, A, 'An incompressible Eulerian formulation for soft solids in fluids', *CTR Annual Research briefs, 2017.*

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# Reference map advection and Level-set reconstruction

Analytical expression for $\phi(\vec{x}, t=0)$ is known.

**Level-set reconstruction**

$$\phi(\vec{x}, t) = \phi(\vec{\xi}, t=0)$$

Reinitialize $\phi$ using Fast-marching method.

**Modified advection equation**

$$\frac{\partial \vec{\xi}(\vec{x}, t)}{\partial t} + H(\psi)\vec{u}.\vec{\nabla}\vec{\xi}(\vec{x}, t) = 0$$

and

$$H(\psi) = \begin{cases} 1 & \Omega_S \\ 0 & else \end{cases}$$

Advantages of this modification:

- removes high frequency content in $\vec{\xi}$ field - simple central schemes.
- robust solver.

[Jain & Mani, *CTR Annual Research Briefs*, 2017]

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# Comparison and updates on the recent approach

| | ***RMT15 (Valkov et al. 2015)*** | ***Present approach*** |
|---|---|---|
| ***Grid*** | Staggered | Collocated |
| ***Nature of the solver*** | Compressible | Incompressible |
| ***Reference map Extrapolation*** | PDE based | Least-square based |
| ***Level-set construction*** | Advected in time | Reconstructed from $t = 0$ (Analytical - exact) |
| ***Smoothing routines*** | Required | No |
| ***Discretization stencil*** | one-sided | central |
| ***Global damping*** | Yes | No |

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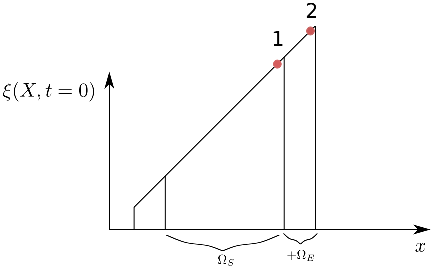
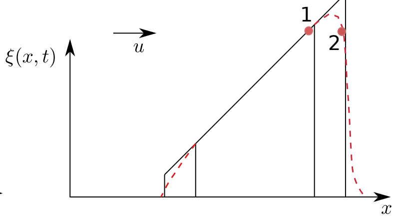
# Robustness due to modified advection equation



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# Solid-Solid contact

Define,

$$\phi_{12} = \frac{\phi^{(1)} - \phi^{(2)}}{2}$$

where $\phi_{12} = 0$ represents a midsurface between two solids.

## Repulsive force

$$\vec{f_{i,j}} = \begin{cases} \gamma_{i,j}\hat{n}_{12i,j} & \phi^{(1)} < 0 \; or \; \phi^{(2)} < 0 \\ 0 & otherwise \end{cases}$$

$$\gamma_{i,j} = k_{rep}\delta_s(\phi_{12i,j})$$

where $\hat{n}_{12i,j}$ is the unit vector normal to contours of $\phi_{12}$ and pointing away from the midsurface, $k_{rep}$ is a prefactor and $\delta_s(x)$ is a compactly supported *influence function*,

$$\delta_s(x) = \begin{cases} \dfrac{1 + cos\,\frac{\pi x}{w_T}}{2w_T} & |x| < -w_T \\ 0 & |x| \geq w_T \end{cases}$$